

Cayley-Hamilton-Newton identities and quasitriangular Hopf algebras

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Abstract

In the framework of the Drinfeld theory of twists in Hopf algebras we construct quantum matrix algebras which generalize the Reflection Equation and the RTT algebras. Finite-dimensional representations of these algebras related to the theory of nonultralocal spin chains are presented. The Cayley-Hamilton-Newton identities are demonstrated. These identities allow to define the quantum spectrum for the quantum matrices. We mention possible applications of the new quantum matrix algebras to constructions of noncommutative analogs of Minkowski space and quantum Poincaré algebras.

1 Twisted Hopf Algebras and related Quantum Matrix Algebras

Consider a quasitriangular Hopf algebra $\mathcal{A}(\Delta, \epsilon, S, \mathcal{R})$ with a universal R -matrix $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$

$$\Delta'(a) = \mathcal{R} \Delta(a) \mathcal{R}^{-1}. \quad (1.1)$$

Let \mathcal{F} be an invertible element of $\mathcal{A} \otimes \mathcal{A}$

$$\mathcal{F} = \sum_i \alpha_i \otimes \beta_i, \quad (\epsilon \otimes id)\mathcal{F} = 1 = (id \otimes \epsilon)\mathcal{F}, \quad \mathcal{F}^{-1} = \sum_i \gamma_i \otimes \delta_i$$

which defines a twist [1] of the algebra \mathcal{A} to a new quasitriangular Hopf algebra $\mathcal{A}^{(F)}$ ($\Delta^{(F)}$, $\epsilon^{(F)}$, $S^{(F)}$, $\mathcal{R}^{(F)}$). Here $\epsilon^{(F)} = \epsilon$, a new coproduct $\Delta^{(F)}$ and a new antipode $S^{(F)}$ are

$$\Delta^{(F)}(a) = \mathcal{F} \Delta(a) \mathcal{F}^{-1}, \quad S^{(F)}(a) = u S(a) u^{-1}, \quad \forall a \in \mathcal{A} \quad (1.2)$$

and a new universal R -matrix is

$$\mathcal{R}^{(F)} = \mathcal{F}^{21} \mathcal{R} \mathcal{F}^{-1}. \quad (1.3)$$

One can derive the following expressions for the element u in (1.2)

$$u = \sum_i \alpha_i S(\beta_i), \quad u^{-1} = \sum_i S(\gamma_i) \delta_i, \quad S(\alpha_i) u^{-1} \beta_i = 1.$$

The coassociativity condition for $\Delta^{(F)}$ is implied by

$$\mathcal{F}^{12} (\Delta \otimes id) \mathcal{F} = \mathcal{F}^{23} (id \otimes \Delta) \mathcal{F}. \quad (1.4)$$

Impose additional relations on \mathcal{F}

$$(\Delta \otimes id) \mathcal{F} = \mathcal{F}^{13} \mathcal{F}^{23}, \quad (id \otimes \Delta) \mathcal{F} = \mathcal{F}^{13} \mathcal{F}^{12}, \quad (1.5)$$

which, together with (1.4), imply the Yang-Baxter equation for \mathcal{F} . Using (1.1) one deduces from (1.5) equations

$$\mathcal{R}^{12} \mathcal{F}^{13} \mathcal{F}^{23} = \mathcal{F}^{23} \mathcal{F}^{13} \mathcal{R}^{12}, \quad \mathcal{F}^{12} \mathcal{F}^{13} \mathcal{R}^{23} = \mathcal{R}^{23} \mathcal{F}^{13} \mathcal{F}^{12}. \quad (1.6)$$

With the Yang-Baxter relations

$$\mathcal{R}^{12} \mathcal{R}^{13} \mathcal{R}^{23} = \mathcal{R}^{23} \mathcal{R}^{13} \mathcal{R}^{12}, \quad \mathcal{F}^{12} \mathcal{F}^{13} \mathcal{F}^{23} = \mathcal{F}^{23} \mathcal{F}^{13} \mathcal{F}^{12}. \quad (1.7)$$

eqs. (1.6) define the twist which is proposed in [2] (we do not assume that $\mathcal{F}^{21} \mathcal{F} = 1 \otimes 1$ as in [2]). We call the pair of $(\mathcal{R}, \mathcal{F})$ *compatible* if eqs (1.6) and (1.7) are satisfied. Many explicit examples of *compatible* pairs are known (see e.g. [3] and references therein). Using relations (1.6), (1.7) one proves an identity

$$\mathcal{R}^{12} (\mathcal{R}^{13} \mathcal{F}^{31}) \mathcal{F}^{21} (\mathcal{R}^{23} \mathcal{F}^{32}) (\mathcal{F}^{21})^{-1} = (\mathcal{R}^{23} \mathcal{F}^{32}) \mathcal{F}^{12} (\mathcal{R}^{13} \mathcal{F}^{31}) (\mathcal{F}^{12})^{-1} \mathcal{R}^{12}. \quad (1.8)$$

Consider the dual Hopf algebra \mathcal{A}^* (the algebra of linear functionals on \mathcal{A}) with generators $\{T_j^i\}$ ($i, j = 1, \dots, n$) and the coproduct $\Delta(T_j^i) = T_k^i \otimes T_j^k$. The matrix representations of the algebra \mathcal{A} in n -dimensional vector space V

is given by $T_j^i(a) = \langle a, T_j^i \rangle \forall a \in \mathcal{A}$ where \langle, \rangle is the pairing between \mathcal{A} and \mathcal{A}^* . Introduce the quantum matrices $L^\pm \in \mathcal{A}$ and $K^\pm \in \mathcal{A}$

$$\begin{aligned} (L^+)_j^i &:= \langle \mathcal{R}, id \otimes T_j^i \rangle, \quad S((L^-)_j^i) := \langle \mathcal{R}, T_j^i \otimes id \rangle, \\ (K^+)_j^i &:= \langle \mathcal{F}, id \otimes T_j^i \rangle, \quad S((K^-)_j^i) := \langle \mathcal{F}, T_j^i \otimes id \rangle, \end{aligned} \quad (1.9)$$

and numerical *compatible* matrices R and F

$$\begin{aligned} R_{12} &= \langle \mathcal{R}, T_1 \otimes T_2 \rangle = \langle L_2^+, T_1 \rangle = \langle S(L_1^-), T_2 \rangle, \\ F_{12} &= \langle \mathcal{F}, T_1 \otimes T_2 \rangle = \langle K_2^+, T_1 \rangle = \langle S(K_1^-), T_2 \rangle. \end{aligned} \quad (1.10)$$

Matrices R_{12} and F_{12} act in $V \otimes V$. Further, we denote the matrices F and R acting in $V_k \otimes V_{k+1}$ by F_k and R_k (here the subscripts k and $k+1$ enumerate different copies of the space V).

Our aim is to define a quantum matrix algebra which generalizes the *RTT* [4] and the Reflection Equation algebras [5]. To this end, define a quantum matrix $M_j^i \in \mathcal{A}$

$$M_j^i := \langle \mathcal{R}^{21} \mathcal{F}^{12}, id \otimes T_j^i \rangle = S((L^-)_k^i) (K^+)_j^k. \quad (1.11)$$

and consider the following pairing

$$\begin{aligned} \langle \mathcal{R}^{21} \mathcal{F}^{12}, id \otimes T_1 T_2 \rangle &= \langle \mathcal{R}^{21} \mathcal{R}^{31} \mathcal{F}^{13} \mathcal{F}^{12}, id \otimes T_1 \otimes T_2 \rangle = \\ S(L_1^-) S(L_2^-) (K_2^+) K_1^+ &= S(L_1^-) S(L_2^-) \hat{F} (K_2^+) K_1^+ \hat{F}^{-1} = \\ S(L_1^-) K_1^+ \hat{F} S(L_1^-) K_1^+ \hat{F}^{-1} &= M_1 \hat{F} M_1 \hat{F}^{-1}. \end{aligned} \quad (1.12)$$

Here $\hat{F} = P_{12} F_{12}$. The pairing of the identity (1.8) with $(T_1 \otimes T_2 \otimes id)$ gives (with the help of (1.12)) the commutation relations for the elements of the matrix M :

$$\hat{R} M_1 \hat{F} M_1 \hat{F}^{-1} = M_1 \hat{F} M_1 \hat{F}^{-1} \hat{R}, \quad (1.13)$$

where $\hat{R} := P_{12} R_{12}$. The algebras $\mathcal{M}(\hat{R}, \hat{F})$ with generators M_j^i and defining relations (1.13) unify the *RTT* ($\hat{F} = P$) and the Reflection Equation ($\hat{F} = \hat{R}$) algebras. The algebras of that kind have been considered in [6] (see also [5], [7]).

The comultiplication acts on M_j^i as follows

$$\Delta(M_j^i) = M_k^l \otimes S((L^-)_l^i) (K^+)_j^k, \quad (1.14)$$

and one can define for the algebra $\mathcal{M}(\hat{R}, \hat{F})$ (1.13) the homomorphic mapping

$$\overline{\Delta}(M_j^i) = M_k^i \widetilde{M}_j^k, \quad \hat{F} M_1 \hat{F}^{-1} \widetilde{M}_1 = \widetilde{M}_1 \hat{F} M_1 \hat{F}^{-1}, \quad (1.15)$$

which is an analog of the braided coproduct [8] (elements \widetilde{M}_j^i generate another copy of the algebra $\mathcal{M}(\hat{R}, \hat{F})$ (1.13)).

The extension of (1.12) on arbitrary products of T 's is straightforward

$$< \mathcal{R}^{21} \mathcal{F}^{12}, id \otimes T_1 T_2 \dots T_n > = M_{\overline{1}} M_{\overline{2}} \dots M_{\overline{n}}, \quad (1.16)$$

where

$$M_{\overline{1}} = M_1, \quad M_{\overline{k+1}} = (\hat{F}_k M_k \hat{F}_k^{-1}). \quad (1.17)$$

Comparing relations (1.12) and (1.13) one can conclude that \mathcal{A}^* is the RTT algebra with defining relations

$$\hat{R} T_1 T_2 = T_1 T_2 \hat{R}, \quad (1.18)$$

if we require that the linear mapping $\phi : \mathcal{A}^* \rightarrow \mathcal{A}$

$$\phi(\alpha) := < \mathcal{R}^{21} \mathcal{F}^{12}, id \otimes \alpha > \in \mathcal{A}, \quad \alpha \in \mathcal{A}^*, \quad (1.19)$$

is nondegenerate: $\phi(\alpha) = 0 \Rightarrow \alpha = 0$.

2 Generalized Cayley-Hamilton-Newton Identities

Consider the case when \hat{R}_{12} (1.10) is a Hecke type matrix

$$\hat{R}^2 = (q - q^{-1}) \hat{R} + I, \quad (2.20)$$

where I is the identity matrix and q is a numerical parameter which is not equal to a root of unity. In this case we have proved [9] that the q -matrix T (1.18) satisfies the generalized Cayley-Hamilton-Newton identities

$$i_q T^{\overline{\wedge i}} = \sum_{k=0}^{i-1} (-1)^{i-k+1} T^{\overline{i-k}} \sigma_k(T), \quad i_q T^{\overline{S i}} = \sum_{k=0}^{i-1} T^{\overline{i-k}} \tau_k(T), \quad (2.21)$$

where $k_q = (q^k - q^{-k})/(q - q^{-1})$,

$$T^{\overline{\wedge i}} := \text{Tr}_{\hat{R}(2\dots k)} \left(A^{(k)} T_1 \dots T_k \right), \quad T^{\overline{S i}} := \text{Tr}_{\hat{R}(2\dots k)} \left(S^{(k)} T_1 \dots T_k \right), \quad (2.22)$$

are versions of k -wedge and k -symmetric powers of q -matrix T [9],

$$\sigma_k(T) := q^k \operatorname{Tr}_{\hat{R}(1\dots k)} \left(A^{(k)} T_1 \dots T_k \right) , \quad \tau_k(T) := q^{-k} \operatorname{Tr}_{\hat{R}(1\dots k)} \left(S^{(k)} T_1 \dots T_k \right) , \quad (2.23)$$

($\sigma_0(T) = \tau_0(T) = 1$) are elementary and complete symmetric functions of the spectrum of the q -matrix T [9],

$$T^{\bar{k}} := \operatorname{Tr}_{\hat{R}(2\dots k)} \left(\hat{R}_1 \dots \hat{R}_{k-1} T_1 \dots T_k \right) , \quad T^{\bar{1}} := T , \quad (2.24)$$

are generalized k -th power of the q -matrix T [10]. Projectors $A^{(k)}$ and $S^{(k)}$ are q -antisymmetrizers and q -symmetrizers respectively

$$A^{(1)} := I , \quad A^{(k)} := \frac{1}{k_q} A^{(k-1)} \left(q^{k-1} - (k-1)_q \hat{R}_{k-1} \right) A^{(k-1)} , \quad (2.25)$$

$$S^{(1)} := I , \quad S^{(k)} := \frac{1}{k_q} S^{(k-1)} \left(q^{1-k} + (k-1)_q \hat{R}_{k-1} \right) S^{(k-1)} . \quad (2.26)$$

The symbol $\operatorname{Tr}_{\hat{R}}$ in (2.22)-(2.24) denotes the quantum trace $\operatorname{Tr}_{\hat{R}}(Z) := \operatorname{Tr}(D Z)$ where a numerical matrix D is defined by the R -matrix

$$D = \operatorname{Tr}_{(2)} \Psi_{12} , \quad \operatorname{Tr}_{(2)}(\Psi_{12} \hat{R}_2) = P_{13} = \operatorname{Tr}_{(2)}(\hat{R}_1 \Psi_{23}) . \quad (2.27)$$

The following properties of matrix D will be important

$$\operatorname{Tr}_{\hat{R}_2} \hat{R}_1 = I_1 , \quad \operatorname{Tr}_{\hat{R}_2}(\hat{R}_1^{\pm 1} Z_1 \hat{R}_1^{\mp 1}) = I_1 \operatorname{Tr}_{\hat{R}}(Z) \Rightarrow \hat{R}_1 D_1 D_2 = D_1 D_2 \hat{R}_1 , \quad (2.28)$$

$$\operatorname{Tr}_{\hat{R}_2}(\hat{F}_1^{\pm 1} Z_1 \hat{F}_1^{\mp 1}) = I_1 \operatorname{Tr}_{\hat{R}}(Z) \Rightarrow \hat{F}_1 D_1 D_2 = D_1 D_2 \hat{F}_1 , \quad (2.29)$$

where Z is an arbitrary quantum ($n \times n$) matrix. One can verify these properties by using matrix versions of the relations (1.6), (1.7). Note that the identities (2.21) and the definitions (2.24) have been written in [9] in a different form with the usual traces instead of the quantum ones. It is clear, however, that these two forms of Cayley-Hamilton-Newton identities are equivalent since they are related by a simple automorphism of the RTT algebra $T \rightarrow D T$.

Now we show that for the case of algebras $\mathcal{M}(\hat{R}, \hat{F})$ (1.13) (unifying the reflection equation and the RTT algebras) one can also prove the identities which are analogous to the generalized Cayley-Hamilton-Newton identities (2.21).

Theorem *The generalized Cayley-Hamilton-Newton identities for the algebra $\mathcal{M}(\hat{R}, \hat{F})$ with defining relations (1.13) have the form*

$$i_q M^{\wedge i} = \sum_{k=0}^{i-1} (-1)^{i-k+1} M^{\overline{i-k}} \sigma_k(M) , \quad i_q M^{S i} = \sum_{k=0}^{i-1} M^{\overline{i-k}} \tau_k(M) , \quad (2.30)$$

where

$$\begin{aligned}
M^{\bar{k}} &:= \text{Tr}_{\hat{R}(2\dots k)} \left(\hat{R}_1 \dots \hat{R}_{k-1} M_{\bar{1}} \dots M_{\bar{k}} \right) , \\
M^{\bar{\wedge} i} &:= \text{Tr}_{\hat{R}(2\dots k)} \left(A^{(k)} M_{\bar{1}} \dots M_{\bar{k}} \right) , \\
M^{\bar{S} i} &:= \text{Tr}_{\hat{R}(2\dots k)} \left(S^{(k)} M_{\bar{1}} \dots M_{\bar{k}} \right) ,
\end{aligned} \tag{2.31}$$

are quantum versions of the k -th, k -wedge and k -symmetric powers of the q -matrix M ,

$$\begin{aligned}
\sigma_k(M) &:= q^k \text{Tr}_{\hat{R}(1\dots k)} \left(A^{(k)} M_{\bar{1}} \dots M_{\bar{k}} \right) , \\
\tau_k(M) &:= q^{-k} \text{Tr}_{\hat{R}(1\dots k)} \left(S^{(k)} M_{\bar{1}} \dots M_{\bar{k}} \right) ,
\end{aligned} \tag{2.32}$$

are elementary and complete symmetric functions of the spectrum of the q -matrix M . The elements $\sigma_k(M)$, $\tau_k(M)$ generate a commutative subalgebra in $\mathcal{M}(\hat{R}, \hat{F})$.

Proof. The identities (2.30) can be obtained directly by the map (1.19) from the RTT identities (2.21). To do this, consider the map

$$\begin{aligned}
&< \mathcal{R}^{21} \mathcal{F}^{12}, id \otimes T^{\bar{i-k}} e_k(T) > = < \mathcal{R}^{21} \mathcal{R}^{31} \mathcal{F}^{13} \mathcal{F}^{12}, id \otimes T^{\bar{i-k}} \otimes e_k(T) > = \\
&= < \mathcal{R}^{21}, id \otimes (T^{\bar{i-k}})_{(1)} > < \mathcal{R}^{31} \mathcal{F}^{13}, id \otimes 1 \otimes e_k(T) > < \mathcal{F}^{12}, id \otimes (T^{\bar{i-k}})_{(2)} > = \\
&= f_{(1)}(S(L^-)) e_k(M) f_{(2)}(K^+) ,
\end{aligned} \tag{2.33}$$

where $e_k(M) = \sigma_k(M)$, $\tau_k(M)$,

$$\Delta(T^{\bar{i-k}}) := (T^{\bar{i-k}})_{(1)} \otimes (T^{\bar{i-k}})_{(2)} .$$

and $f_{(1,2)}(\cdot)$ are some functions explicit forms of which are not important. We need only the relation $[e_k(M), f_{(2)}(K^+)] = 0$ which is deduced from the relation $[e_k(M), K^+] = 0$. Indeed,

$$\begin{aligned}
&\text{Tr}_{\hat{R}(1,\dots,k)} (A^{(1,k)} M_{\bar{1}} \dots M_{\bar{k}}) K_0^+ = \\
&K_0^+ \text{Tr}_{\hat{R}(1,\dots,k)} (\hat{F}_{0 \rightarrow k-1} A^{(0,k-1)} M_{\bar{0}} \dots M_{\bar{k-1}} \hat{F}_{0 \rightarrow k-1}^{-1}) \\
&= K_0^+ \text{Tr}_{\hat{R}(0,\dots,k-1)} (A^{(0,k-1)} M_{\bar{0}} \dots M_{\bar{k-1}}) = K_0^+ e_k(M) .
\end{aligned} \tag{2.34}$$

Here we have used $M_1 K_0^+ = K_0^+ \hat{F}_0 M_0 \hat{F}_0^{-1}$.

Using (2.34) one can rewrite (2.33) in the form

$$< \mathcal{R}^{21} \mathcal{F}^{12}, id \otimes T^{\bar{i-k}} e_k(T) > = < \mathcal{R}^{21} \mathcal{F}^{12}, id \otimes T^{\bar{i-k}} > e_k(M) = M^{\bar{i-k}} e_k(M) . \tag{2.35}$$

Now it is evident that identities (2.21) and (2.30) are related via the map (1.16).

The commutativity of the elements $\sigma_k(M)$ and $\tau_k(M)$ is proved by the method presented in [11].

3 Automorphisms for the algebras $\mathcal{M}(\hat{R}, \hat{F})$.

In the paper [11] we have obtained the quantum Cayley-Hamilton-Newton identities for the algebra of q -matrices M_j^i with different defining relations (cf. with (1.13))

$$\hat{R} M_1 \hat{F} M_1 \hat{F}^{-1} = M_1 \hat{F} M_1 \hat{F}^{-1} \hat{R}^{\hat{F}\hat{F}}. \quad (3.36)$$

where \hat{R} is a Hecke type R -matrix, the pair (\hat{R}, \hat{F}) is *compatible* and [11]

$$\hat{R}^{\hat{F}\hat{F}} := \hat{F}^2 \hat{R} \hat{F}^{-2} = D'_1 D'_2 \hat{R} (D'_1 D'_2)^{-1}. \quad (3.37)$$

Here a numerical matrix D' is an analog of the matrix D (2.27):

$$D' = \text{Tr}_{(2)} \Phi_{12}, \quad \text{Tr}_{(2)}(\Phi_{12} \hat{F}_2) = P_{13} = \text{Tr}_{(2)}(\hat{F}_1 \Phi_{23}), \quad (3.38)$$

but is related to the matrix \hat{F} .

The remarkable fact is that these two algebras $\mathcal{M}(\hat{R}, \hat{F})$ (1.13) and (3.36) are connected by a simple transformation. To manifest this connection we need to introduce an algebra which unifies the both algebras (1.13) and (3.36). This algebra is defined by relations

$$\hat{R} M_1 \hat{A} M_1 (\hat{F}^{-1})^X = M_1 \hat{A} M_1 (\hat{F}^{-1})^X \hat{R}^X. \quad (3.39)$$

where $(\hat{F}^{-1})^X = X_1 X_2 \hat{F}^{-1} (X_1 X_2)^{-1}$, $\hat{R}^X = X_1 X_2 \hat{R} (X_1 X_2)^{-1}$, X_j^i is an invertible numerical $(n \times n)$ matrix, the pair (\hat{R}, \hat{F}) is *compatible* and matrix \hat{A} is defined by the matrix Φ , the skew inverse of the matrix \hat{F} (3.38)

$$\hat{A}_{12} := X_2 \Phi_{21}^{-1} X_1^{-1}. \quad (3.40)$$

Thus, the algebra (3.39) is defined by the *compatible* pair (\hat{R}, \hat{F}) and the invertible matrix X . For a Hecke-type \hat{R} and an arbitrary X one can prove, using the methods of the paper [11], the Cayley-Hamilton-Newton identities similar to (2.30) for the algebra (3.39).

For $X = D$ we find that $(\hat{F}^{-1})^X = \hat{F}^{-1}$, $\hat{R}^X = \hat{R}$, $\hat{A} = \hat{F}$ (these equalities can be obtained from (2.29) and (3.40)) and therefore the algebra (3.39) coincides with the algebra $\mathcal{M}(\hat{R}, \hat{F})$ (1.13).

On another hand, for $X = D'$ we have $(\hat{F}^{-1})^X = \hat{F}^{-1}$, $\hat{R}^X = \hat{R}^{\hat{F}\hat{F}}$ (3.37), $\hat{A} = \hat{F}$ and, thus, the algebra (3.39) converts to the algebra (3.36).

Now we note that relations (3.39) are invariant under the transformations

$$\hat{R} \longrightarrow U_1 U_2 \hat{R} (U_1 U_2)^{-1}, \quad \hat{F} \longrightarrow U_1 U_2 \hat{F} (U_1 U_2)^{-1}, \quad (3.41)$$

$$\hat{A} \longrightarrow Y_1 U_2 \hat{A} (U_1 Y_2)^{-1}, \quad M \longrightarrow U M Y^{-1}, \quad X \longrightarrow Y X U^{-1},$$

where U and Y are invertible numerical $(n \times n)$ matrices. One can always find such matrices U, Y that $X = D \rightarrow X = D'$ and therefore relate the algebras of the types (1.13) and (3.36) by transformations (3.41). So, using (3.41) one can obtain the Cayley-Hamilton-Newton identities for one algebra from another and vice versa.

4 Discussion

1. The generators M_j^i of the matrix algebra $\mathcal{M}(\hat{R}, \hat{F})$ (1.13) have a natural finite-dimensional matrix representation

$$(M_j^i)_m^n = (\hat{R} \hat{F})_{mj}^{ni},$$

which is dictated by the definition (1.11). This representation and the braided coproduct $\overline{\Delta}(M_j^i)$ (1.15) give a possibility to define the integrable nonultralocal spin chains following the approach of the paper [6].

2. The explicit form (1.14) for the comultiplication $\Delta(M_j^i)$ leads to the conclusion that the algebra (3.39) (and, hence, its particular formulations (1.13) and (3.36)) are covariant under the quantum transformations

$$M_j^i \rightarrow (T M \tilde{T}^{-1})_j^i \equiv M_l^k \otimes T_k^i S(\tilde{T}_j^l) \quad (4.42)$$

where elements $\{T_j^i, \tilde{T}_l^k\}$ generate the algebra

$$\begin{aligned} \hat{R} T_1 T_2 &= T_1 T_2 \hat{R}, \quad \hat{A} T_1 \tilde{T}_2 = \tilde{T}_1 T_2 \hat{A}, \\ \hat{R}^X \tilde{T}_1 \tilde{T}_2 &= \tilde{T}_1 \tilde{T}_2 \hat{R}^X, \quad \hat{F}^X \tilde{T}_1 \tilde{T}_2 = \tilde{T}_1 \tilde{T}_2 \hat{F}^X. \end{aligned} \quad (4.43)$$

3. There exists a Heisenberg double of the $\mathcal{M}(\hat{R}, \hat{F})$ (3.39) and the RTT

algebras:

$$\begin{aligned}\hat{R} M_1 \hat{A} M_1 \hat{B} &= M_1 \hat{A} M_1 \hat{B} \hat{R} , \\ T_1 M_2 &= \hat{A} M_1 \hat{B} T_1 , \\ \hat{R} T_1 T_2 &= T_1 T_2 \hat{R} .\end{aligned}\tag{4.44}$$

where we rewrite the relations (3.39) using $\hat{B} = X_1 X_2 \hat{F}^{-1}$. For this Heisenberg double one can define the analog of Alekseev-Faddeev discrete evolution (see [12]) $M \rightarrow M$, $T \rightarrow T^{(k)} = M^k T$.

4. One can define the de Rham differential complex over the algebra $\mathcal{M}(\hat{R}, \hat{F})$ (3.39) (for matrix \hat{R} of Hecke type)

$$\begin{aligned}\hat{R} M_1 \hat{A} M_1 \hat{B} &= M_1 \hat{A} M_1 \hat{B} \hat{R} , \\ \hat{R} (dM_1) \hat{A} M_1 \hat{B} &= M_1 \hat{A} (dM_1) \hat{B} \hat{R}^{-1} , \\ \hat{R} (dM_1) \hat{A} (dM_1) \hat{B} &= -(dM_1) \hat{A} (dM_1) \hat{B} \hat{R}^{-1} .\end{aligned}\tag{4.45}$$

5. The construction of the Heisenberg double (4.44) and the covariance (4.42) give a possibility to apply the algebra $\mathcal{M}(\hat{R}, \hat{F})$ (3.36) in the (2×2) case to the theory of a quantum Minkowski space and a quantum Poincaré algebra [13], [14]. We note here that this subject in the framework of twist theory has been already discussed [15] in different context.

6. The algebra $\mathcal{M}(\hat{R}, \hat{F})$ in the form (3.36) for the dynamical \hat{R} and \hat{F} matrices has been used for R -matrix quantization of the elliptic Ruijsenaars-Schneider model [16].

Acknowledgements: We are indebted to L.Hlavaty, P. Kulish and S.Pakuliak for valuable discussions. This work was supported in parts by CNRS grant PICS No. 608 and RFBR grant No. 98-01-2033. The work of AI and PP was also supported by the RFBR grant No. 97-01-01041.

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